Linear Homogenous Second-Order Differential Equations

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Seminar in Engineering Analysis

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Outline

- Review last class
- Second-order linear differential equations
  - Homogenous equations with constant coefficients: y'' + ay' + by = 0
  - General solution has three different forms depending on $a^2 - 4b$
  - Applications to spring-mass systems
    - Overdamped and underdamped systems

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Review definitions

- Ordinary versus partial
- Linear versus nonlinear
- Order of the equation
- Homogenous versus non-homogenous
- Physical problems: rate equations and Newton's second law $md^2y/dt^2 = F$
- General versus particular solutions

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Review ODE Examples

- Third-order, linear, homogenous $$\frac{d^3y}{dx^3} + \sin(x)\frac{dy}{dx} - x^2y = 0$$
- Second-order, non-linear, homogenous $$\frac{d^2y}{dx^2} + \sin(y) = 0$$
- Second-order, linear, non-homogenous $$\frac{d^2y}{dx^2} + y = e^x\cos(x)$$
- Third-order, non-linear, non-homogenous $$\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 1$$

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Review Separable Forms

- Simple differential equations can be written as integrals
  - Even if numerical quadrature is required this is more accurate than numerical solution of ODE

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y = \int f(x)dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int g(y)dy = \int f(x)dx + C$$

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right) \quad \Rightarrow \quad \int\frac{dx}{x} = \int\frac{du}{h(u) - u} + C$$

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Review Exact Forms

- Differential equation of the form P(x,y)dx + Q(x,y)dy = 0 is called an exact form if $\partial P/\partial y = \partial Q/\partial x$
- Special solution process available in this case
- May be able to find find integrating factors if P(x,y)dx + Q(x,y)dy = 0 is not exact

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Review First-order Equations

- First order rate equation where rate is proportional to amount dy/dt = -ky
- $y = y_0 e^{-k(t-t_0)}$
- General linear first order equation for y(x): dy/dx + f(x)y = g(x) has closed form solution shown below
- C is found from initial condition

$$p = \int f(x)dx \qquad y = e^{-p}\left[C + \int e^p g(x)dx\right]$$

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Review Existence/Uniqueness

- Important because we can try numerical solution of an ODE with no solution
- Examine dy/dx = f(x,y) with $y(x_0) = y_0$ in a region $|x - x_0| < a$ and $|y - y_0| < b$
- Derivate is bounded: $|f(x,y)| \leq K$
- Equation has a solution in region $|x - x_0| < \min(a, b/K)$
- Uniqueness requires $|\partial f/\partial y| \leq M$

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Review Second Order ODEs

- Most general requires numerical solution $\phi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0$
- Linear non-homogenous $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$
- Linear homogenous $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$
- Constant coefficient $\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$

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Review Linear Homogenous

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

- A basis of solutions for this equation is any two linearly independent solutions y_1 and y_2
- $y = c_1y_1 + c_2y_2$ is a general solution where c_1 and c_2 can be used to fit initial or boundary conditions
  - Initial conditions specify y(0) and y'(0)
  - Boundary conditions specify y(a) and y(b)

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Review Constant Coefficients

- Seek solution to

$$c\frac{d^2y}{dx^2} + d\frac{dy}{dx} + ey = 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$

- Solution is $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$$

- Verified solution by substitution into differential equation

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Special Cases

- What if solutions λ_1 and λ_2 are a double root ($\lambda_1 = \lambda_2$) or complex roots?
- Complex roots: $\omega^2 = \beta - (\alpha/2)^2 > 0$
  - give two linearly independent solutions
  - Can rewrite in terms of sines and cosines
- Double root: $\lambda_1 = \lambda_2 = -\alpha/2$
  - gives only one linearly independent solution
  - Must find another solution for basis

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Double Root

- Given y_1, use method called reduction of order to find second solution y_2
- Let $y_2 = uy_1$ so that $y_2' = u' y_1 + uy_1'$ and $y_2'' = u'' y_1 + 2 u' y_1' + u y_1''$
- Substitute into $y_2'' + \alpha y_2' + \beta y_2 = 0$
- Result, $u'' y_1 + 2 u' y_1' + u y_1'' + \alpha[u' y_1 + uy_1'] + \beta uy_1 = 0$, is rearranged below
- $u[y_1'' + \alpha y_1' + \beta y_1] + u'' y_1 + 2 u' y_1' + \alpha u' y_1 = 0$

Equals zero because y_1 is a solution of the ODE

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Double Root II

- Solve: $u'' y_1 + 2 u' y_1' + \alpha u' y_1 = 0$
- Substitute $y_1 = e^{\lambda x} = e^{-\alpha x/2}$ and define $U = u'$ so that $y_1' = (-\alpha/2)e^{-\alpha x/2}$ and $u'' = U'$
- $U' e^{-\alpha x/2} + 2U (-\alpha/2) e^{-\alpha x/2} + \alpha U e^{-\alpha x/2} = 0$
- After division by $e^{-\alpha x/2}$ result is $U' + (-\alpha + \alpha)U = U' = u'' = 0$
- If $u'' = 0$, $u' = C_1$ and $u = C_1x + C_2$ so $y_2 = uy_1 = (C_1x + C_2)y_1 = (C_1x + C_2) e^{-\alpha x/2}$

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Double Root III

- Results for double root are $y_1 = e^{-\alpha x/2}$ and $y_2 = (C_1x + C_2) e^{-\alpha x/2}$
- These are two linearly-independent solutions to our ODE
  - Can use any linear combination of y_1 and y_2
  - Take C_2y_1 and $y_2 - C_2y_1$ as solutions
  - Gives $C_2e^{-\alpha x/2}$ and $C_1xe^{-\alpha x/2}$ as the two linearly independent solutions
  - General solution is $(C_1x + C_2) e^{-\alpha x/2}$

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Complex Roots

- Still have linearly independent general case solutions: $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$
- Complex roots are $-\alpha/2 \pm i\omega$ where $i^2 = -1$ and $\omega^2 = \beta - (\alpha/2)^2 > 0$
- Get solution with more insight by using Euler formula for complex exponential
- $e^{ix} = \cos x + i \sin x$
- $e^{-ix} = \cos x - i \sin x$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = \frac{-\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \beta}$$

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Complex Roots II

- General solution: $y = C_1y_1 + C_2 y_2$ with $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ and $\lambda = -\alpha/2 \pm i\omega$
- $y = C_1 e^{(-\alpha x/2 + i\omega)x} + C_2 e^{(-\alpha x/2 - i\omega)x}$
- $y = e^{-\alpha x/2} [C_1e^{i\omega x} + C_2 e^{-i\omega x}]$
- Apply Euler formula to $e^{i\omega x}$ and $e^{-i\omega x}$
- $y = e^{-\alpha x/2} [C_1(\cos \omega x + i \sin \omega x) + C_2 (\cos \omega x - i \sin \omega x)] = e^{-\alpha x/2} [(C_1 + C_2)\cos \omega x + i(C_1 - C_2)\sin \omega x] = e^{-\alpha x/2} [A\cos \omega x + B\sin \omega x]$

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Solutions to $y'' + \alpha y' + \beta = 0$

- Three cases depending on $\omega^2 = \beta - \alpha^2/4$
- Double root when $\beta = \alpha^2/4$:
  - $y = (C_1x + C_2) e^{-\alpha x/2}$
- Complex roots when $\beta > \alpha^2/4$
  - $y = e^{-\alpha x/2} [A\cos \omega x + B\sin \omega x]$
- Distinct real roots when $\beta < \alpha^2/4$
  - $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = \frac{-\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \beta} = -\frac{\alpha}{2}\left(1 \mp \sqrt{1 - \frac{4\beta}{\alpha^2}}\right)$$

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Solving y'' + αy' + b = 0

- First determine which solution you have based on α and β values
  - $\beta = \alpha^2/2$: $y = (C_1x + C_2)\, e^{-\alpha x/2}$
  - $\beta > \alpha^2/2$: $y = e^{-\alpha x/2}\,[A\cos \omega x + B\sin \omega x]$
  - $\beta < \alpha^2/2$: $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
- Compute ω or as λ_1 and λ_2 as required
- Use initial or boundary conditions to evaluate constants

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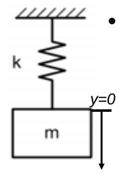
Example: Mechanical Systems

- Simplest system is a spring mass system where the Hooke's law force is the spring constant times the displacement of the spring from equilibrium



- Choose a y coordinate as follows
  - y is zero when the spring-mass system is in static equilibrium
  - as y increases, spring length increases and a force, F = -ky, acts on the mass

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Mechanical Systems II

- From Newton's second law, F = ma, we obtain $md^2y/dx^2 = -ky$; k units are N/m
  - In standard form $d^2y/dx^2 + (k/m)y = 0$; k/m has units of N/(m·kg) = ~~kg·m~~/s^2 /(~~m·kg~~) = s^{-2}
  - Here $\alpha = 0$ and $\beta = k/m > \alpha^2$ so we have complex roots with $\omega^2 = \beta - \alpha^2/2 = k/m - 0$
  - Solution is $e^0[A\sin \omega t + B\cos \omega t]$
  - Initial conditions determine constants: $B = y_0 = y(0)$ and $A = v_0/\omega = y'(0)/\omega$
  - $y = (y'(0)/\omega) \sin \omega t + y(0) \cos \omega t$

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Alternative Formulations

- Can make solution dimensionless by dividing by initial displacement, y_0
- $y/y_0 = v_0/(\omega y_0)\sin \omega t + \cos \omega t$, where $\omega = (k/m)^{1/2}$ has correct units of s^{-1}
- Plot shown on next chart
- Can also write solution in terms of only the cosine: $y = C \cos(\omega t - \delta)$
  - Details shown on chart after next

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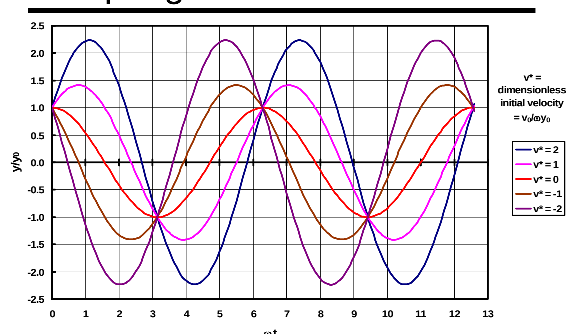
Spring-mass Oscillations



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Alternative Formulations

- Proposed alternative with equation from trigonometry for cos(a – b)

$$y = C\cos(\omega t - \delta) = C\sin\delta\sin\omega t + C\cos\delta\cos\omega t$$

- How does this match original equation?

$$y = A\sin\omega t + B\cos\omega t$$

- Matches if $A = C \sin \delta$ and $B = C \cos \delta$
- This gives $A^2 + B^2 = C^2 \sin^2 \delta + C^2 \cos^2 \delta = C^2$ and $A/B = (C \sin \delta)/(C \cos \delta) = \tan \delta$ or $\delta = \tan^{-1}(A/B) = \tan^{-1}(v_0/y_0\, \omega)$

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Add Damping

- A damping (friction) force is assumed to be proportional to velocity, y' = dy/dt
- Damping coefficient, c, gives F = -c y'
- c units are N·s/m = kg·m/s^2 ·(s/m)= kg/s
- Minus sign is used because force retards motion
- Newton's second law now becomes md^{2}y/dt^2 = –ky – cdy/dt

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Solutions with Damping

- Standard form for with damping is d^{2}y/dt^2 + (c/m)dy/dt + (k/m)y = 0
- This is constant coefficient equation with α = c/m and β = k/m
- $\alpha^2 - 4\beta = (c/m)^2 - 4$ k/m
- k/m has units of s^{-2}
- c/m units are (kg/s)/kg = s^{-1}, consistent with calculation of (c/m)^2 - 4 k/m

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Solutions with Damping II

- d^{2}y/dt^2 + (c/m)dy/dt + (k/m)y = 0
- d^{2}y/dt^2 + αdy/dt + βy = 0

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = -\frac{c}{2m}\left(1 \mp \sqrt{1 - \frac{4km}{c^2}}\right)$$

- If 4 km/c^2 = 1, we have a double root with solution y = (C$_1$ + C$_2$t)e$^{-ct/2m}$
- If 4 km/c^2 > 1, complex roots give solution y = e$^{-ct/2m}$ (A sin ωt + B cos ωt)

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \frac{c}{2m}\sqrt{\frac{4km}{c^2} - 1}$$

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Solutions with Damping III

- d^{2}y/dt^2 + (c/m)dy/dt + (k/m)y = 0
- d^{2}y/dt^2 + αdy/dt + βy = 0

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = -\frac{c}{2m}\left(1 \mp \sqrt{1 - \frac{4km}{c^2}}\right)$$

- 4 km/c^2 < 1 gives two real roots (both negative) in solution y = $C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$
- Look at names for each solution and alternative expression of conditions

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Solutions with Damping IV

- Conditions for solutions comparing 4km/c^2 to 1 can be written in terms of c^2 and 4km or (c/m)^2 and 4 k/m
- Double root occurs when c^2 = 4km is called critical damping
- Complex roots occur when c^2 < 4km is called under damping
- Two distinct real roots occur when c^2 > 4km is called over damping
- Look at each solution with initial conditions y(0) = y_0 and y'(0) = v_0

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Over Damping

- Two real roots, both negative
- $y = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$
- $y' = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t}$
- $y_0 = y(0) = C_1e^{\lambda_1 0} + C_2e^{\lambda_2 0} = C_1 + C_2$
- $v_0 = y'(0) = C_1\lambda_1e^{\lambda_1 0} + C_2\lambda_2e^{\lambda_2 0} = C_1\lambda_1 + C_2\lambda_2$; solve two equations for C_1 and C_2
- $C_1 = (\lambda_2 y_0 - v_0)/(\lambda_2 - \lambda_1)$
- $C_2 = (v_0 - \lambda_1 y_0)/(\lambda_2 - \lambda_1)$

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Get Dimensionless Parameters

- Divide numerator and denominator of C_1 and C_2 by λ_1
- Substitute result for C_1 and C_2 into original equation
- Divide both sides of equation by y_0

$$\frac{y}{y_0} = \frac{\frac{\lambda_2}{\lambda_1} - \frac{v_0}{y_0\lambda_1}}{\frac{\lambda_2}{\lambda_1} - 1} e^{\lambda_1 t} + \frac{\frac{v_0}{y_0\lambda_1} - 1}{\frac{\lambda_2}{\lambda_1} - 1} e^{\lambda_2 t}$$

$$\lambda t = -\frac{ct}{2m}\left(1 \mp \sqrt{1 - \frac{4km}{c^2}}\right)$$

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Over Damping Results

- Take λ_2 to be larger solution
- Coefficients depend on km/c^2 and mv_0/cy_0

$$\frac{\lambda_2}{\lambda_1} = \frac{-\frac{c}{2m}\left(1 + \sqrt{1 - \frac{4km}{c^2}}\right)}{-\frac{c}{2m}\left(1 - \sqrt{1 - \frac{4km}{c^2}}\right)} = \frac{1 + \sqrt{1 - \frac{4km}{c^2}}}{1 - \sqrt{1 - \frac{4km}{c^2}}}$$

$$\frac{v_0}{y_0\lambda_1} = \frac{v_0}{-\frac{cy_0}{2m}\left(1 - \sqrt{1 - \frac{4km}{c^2}}\right)} = \frac{-2\frac{mv_0}{cy_0}}{\left(1 - \sqrt{1 - \frac{4km}{c^2}}\right)}$$

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Over Damping Results II

- Solutions for y/y_0 depend on three parameters: ct/m, km/c^2 and mv_0/cy_0
- ct/m is dimensionless time, but exponential decay, $e^{\lambda t}$, has λt exponent of $-ct[1 \pm (1 - 4km/c^2)^{1/2}]/2m$ that depends on km/c^2
- Look at plots of y/y_0 versus ct/m with mv_0/cy_0 as a parameter
- Different charts for different damping measured as km/c^2

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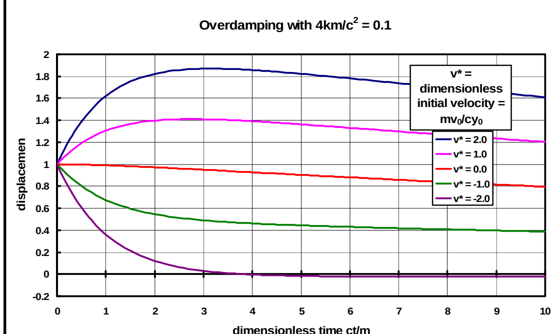

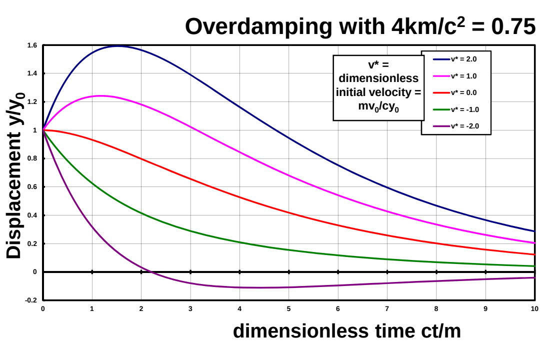

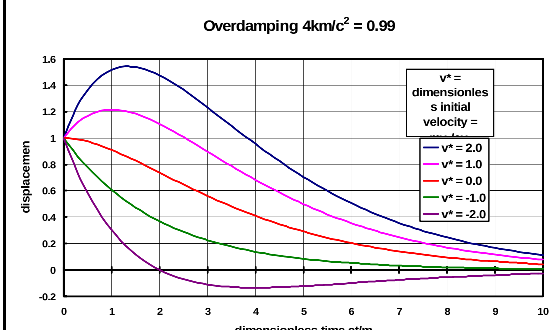

Critical Damping ($c^2 = 4km$)

- Double root: $y_0 = (C_1 + C_2 0)e^{-c0/2m} = C_1$
- $y' = (y_0 + C_2 t)(-c/2m)e^{-ct/2m} + C_2 e^{-ct/2m}$
- $v_0 = y'(0) = (-c\, y_0 /2m) + C_2$
- $y = [y_0 + (v_0 + c\, y_0/2m)t]e^{-ct/2m}$
- Boundary between over and under damping, y/y_0 depends on ct/m and mv_0/cy_0
- $y/y_0 = [1 + v_0 t/y_0 + ct/2m]e^{-ct/2m}$
- $y/y_0 = [1 + (2mv_0/cy_0 + 1)ct/2m]e^{-ct/2m}$

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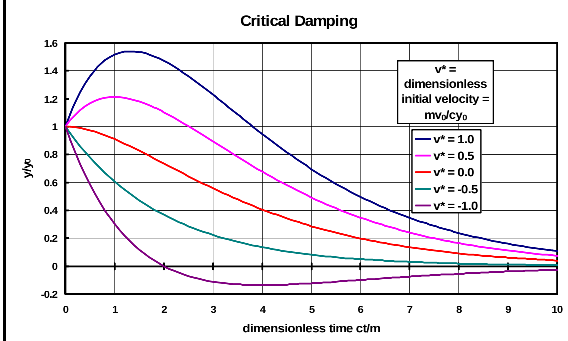

Under Damping

- $y = e^{-ct/2m}[A \sin \omega t + B \cos \omega t]$
- $y' = (-c/2m)e^{-ct/2m}[A \sin \omega t + y_0 \cos \omega t] + e^{-ct/2m} [A \cos \omega t - B \sin \omega t]\, \omega$
- Initial conditions: $y(0) = y_0$ and $y'(0) = v_0$
- $y_0 = e^{-c0/2m}[A \sin 0 + B \cos 0] = B$
- $v_0 = y'(0) = (-c/2m)e^{-c0/2m} [A \sin 0 + B \cos 0] + e^{-c0/2m} [A \cos 0 - B \sin 0]\, \omega$
- $v_0 = (-c/2m)B + \omega A$
- Results: $B = y_0$ and $A = (v_0 + c\, y_0/2m)/\omega$

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Under Damping II

- Divide by y_0 for equation in y/y_0
- $y/y_0 = e^{-ct/2m}[(v_0/y_0 + c/2m)(\sin \omega t)/\omega + \cos \omega t]$
- $\omega = (c/2m)(4km/c^2 – 1)^{1/2}$
- $\omega t = (ct/2m)(4km/c^2 – 1)^{1/2}$
- $(v_0/y_0 + c/2m)/\omega = (v_0/y_0 + c/2m) / (c/2m) / (4km/c^2 - 1)^{1/2} = (2mv_0/cy_0 + 1) / (4km/c^2 - 1)^{1/2}$
- y/y_0 depends on ct/m, km/c^2, and mv_0/cy_0

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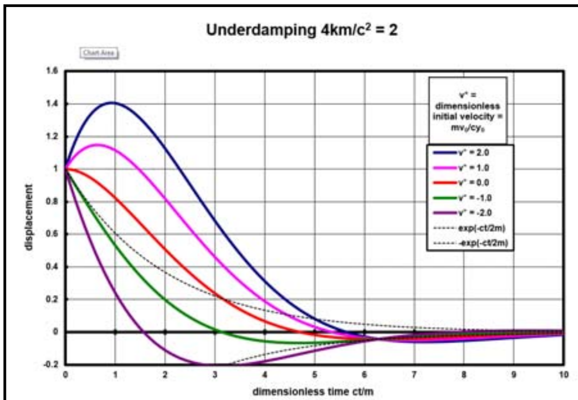

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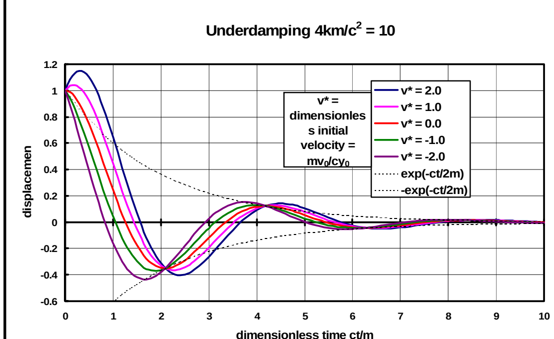
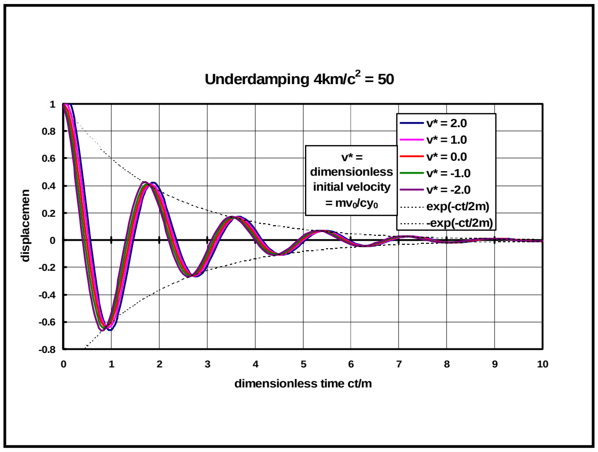
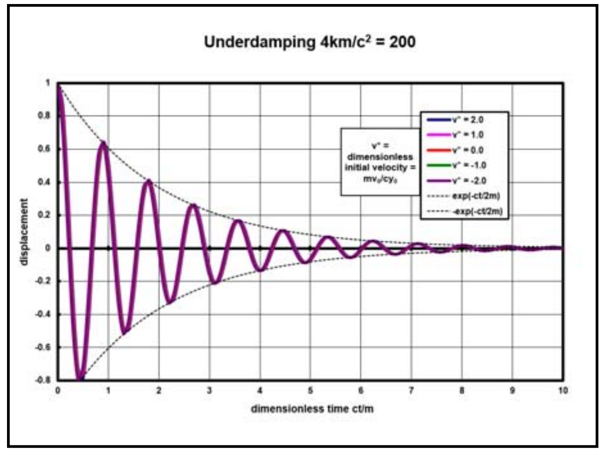



Underdamping 4km/c² = 50
displacemen
dimensionless time ct/m
v* = dimensionless initial velocity = mv₀/cy₀
v* = 2.0
v* = 1.0
v* = 0.0
v* = -1.0
v* = -2.0
exp(-ct/2m)
-exp(-ct/2m)


Underdamping 4km/c² = 200
displacement
dimensionless time ct/m